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Journal of Algebra 308 (2007) 641–653
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Nilpotency of p-complements and p-regular conjugacy class sizes ☆

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Received 8 March 2006
Available online 13 November 2006
Communicated by Evgenii Khukhro

Abstract

Let G be a finite p-solvable group. We prove that if the set of conjugacy class sizes of all p'-elements of G is $\{1, m, p^a, mp^a\}$, where m is a positive integer not divisible by p, then the p-complements of G are nilpotent and m is a prime power. This result partially extends a theorem for ordinary classes which asserts that if the set of conjugacy class sizes of a finite group G is exactly $\{1, m, n, mn\}$ and $(m, n) = 1$, then G is nilpotent.
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Keywords: Finite groups; Conjugacy classes; p-Regular elements

1. Introduction

We will assume that any group is finite. For a prime p and a p-solvable group G, we denote by $G_{p'}$ the set of p-regular elements (or p'-elements) of G and by $\mathrm{Con}(G_{p'})$ the set of the p-regular conjugacy classes. We will call the index of $x \in G$ the size of its conjugacy class, x^G.

Recently, there has been some interest in studying the p-structure of groups (or p-solvable groups) from the conjugacy class sizes of its p-regular elements. This study cannot often be transferred directly from techniques used in ordinary classes because when G is not p-nilpotent, then the conjugacy class sizes of any p-complement of G may not divide the corresponding

☆ This paper was partially supported by grant MTM20004-06067-C02-02 and by grant GV05/265.
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doi:10.1016/j.jalgebra.2006.10.007

conjugacy class sizes in the whole group. For instance, it is well known that if all the conjugacy class sizes of a group G are powers of a fixed prime q then G is nilpotent. When all the conjugacy class sizes of p'-elements in G consist of powers of a fixed prime equal or distinct from p, then the p-complements of G are nilpotent too. However, if $\pi = \{p, q\}$ with $q \neq p$ and the class sizes of p'-elements of G are π-numbers, then the p-complements of G are not necessarily nilpotent, even though the group is p-solvable. It is easy to find examples of p-solvable groups whose class sizes of p'-elements are $\{1, p^a, p^a q^b\}$ and such that the p-complements of G are not nilpotent. On the other hand, a classical theorem by A. Camina (see [6] for instance) asserts that if G is a group whose conjugacy class sizes are $\{1, r^a, q^b, r^a q^b\}$, with q and r primes and a and b positive integers, then G is nilpotent. In [4] and [5] the authors extended Camina's theorem by showing that if the conjugacy class sizes of a group are $\{1, m, n, mn\}$ for two arbitrary coprime numbers m and n, then the group is nilpotent. The aim of this note is to obtain arithmetical conditions on the p-regular conjugacy class sizes which guarantee the nilpotency of the p-complements. Our main result is the following

Theorem A. *Let G be a p-solvable group whose conjugacy class sizes of p'-elements are $\{1, m, p^a, mp^a\}$, where m is an integer not divisible by p. Then the p-complements of G are nilpotent and $m = q^b$ for some prime q distinct from p.*

We believe that this could probably be a starting point for a proof of a more general result, that is, to show the nilpotency of the p-complements of a group when the set of its p-regular conjugacy class sizes is $\{1, m, n, mn\}$ where m and n are arbitrary coprime numbers.

2. A particular case of Theorem A

We are going to prove a particular case of the main theorem: when m is a prime power. We will need some well-known elementary results on conjugacy classes.

Lemma 1. *Suppose that G is a finite group and let p be a prime such that every conjugacy class size of $G_{p'}$ is a p'-number. Then $G = P \times H$ where P is a Sylow p-subgroup and H is a p-complement of G.*

Proof. This is Lemma 1 of [6]. □

Lemma 2. *Let G be a finite group. A prime p does not divide any conjugacy class size of G if and only if G has a central Sylow p-subgroup.*

Proof. See for instance Theorem 33.4 of [7]. □

We will make use of the classic Thompson's $A \times B$-Lemma.

Theorem 3. *Let AB be a finite group represented as a group of automorphisms of a p-group G with $[A, B] = 1 = [A, C_G(B)]$, B a p-group and $A = \mathbf{O}^p(A)$. Then $[A, G] = 1$.*

Proof. See for instance 24.2 of [1]. □

We also need the following result. We denote by π an arbitrary set of primes.

Lemma 4. *Let G be a π-separable group.*

(a) *Let $x \in G$ such that $|x^G|$ is a π-number. Then $x \in \mathbf{O}_{\pi,\pi'}(G)$.*
(b) *The conjugacy class size of any π'-element in G is a π-number if and only if G has abelian Hall π'-subgroups. In this case, $l_{\pi'}(G) \leqslant 1$.*

Proof. Part (a) is Theorem C of [3] and part (b) is Lemma 7 of [3]. $\square$

From Lemma 2, it follows that when every conjugacy class size of a group G is a power of some fixed prime q, then G is nilpotent. We could ask if whenever G is a p-solvable group and the conjugacy class sizes of $G_{p'}$ are $\{p, q\}$-numbers for some prime $q \neq p$, then the p-complements of G need to be nilpotent. But this is certainly false as we have pointed out in the introduction. Nevertheless, we show in the next result that each p-complement of G has a normal Sylow q-subgroup.

Theorem 5. *Let G be a finite p-solvable group and $\pi = \{p, q\}$ with q and p two primes. Suppose that the sizes of the conjugacy classes of $G_{p'}$ are π-numbers. Then G is solvable, it has abelian π-complements and every p-complement of G has a normal Sylow q-subgroup.*

Proof. We argue by induction on $|G|$. We will prove first that G is solvable. Assume that $\mathbf{O}^p(G) < G$. As the hypothesis is inherited by normal subgroups, it follows by induction that $\mathbf{O}^p(G)$ is solvable and hence G is solvable too. Thus, we may suppose that $\mathbf{O}^{p'}(G) < G$ and use bars to work in $\overline{G} = G/\mathbf{O}^{p'}(G)$. Notice that for any $\bar{x} \in \overline{G}$, we may assume that $x \in G_{p'}$ and as $|\bar{x}^{\overline{G}}|$ divides $|x^G|$ and $|x^G|$ is a π-number, it follows that $|\bar{x}^{\overline{G}}|$ is a q-number. Hence, by applying Lemma 2, we obtain that $\overline{G}$ is nilpotent. As $\mathbf{O}^{p'}(G)$ is solvable by induction, we conclude that G is solvable too.

We show now that every p-complement of G has a normal Sylow q-subgroup. If y is a π'-element of G, then in particular $y \in G_{p'}$ so by hypothesis $|y^G|$ is a π-number. By Lemma 4(b), we have that G has abelian Hall π'-subgroups and $l_{\pi'}(G) \leqslant 1$. Let T be a π-complement of G, so $\mathbf{O}_\pi(G)T \trianglelefteq G$. Suppose first that $\mathbf{O}_\pi(G) = 1$, so $T = \mathbf{O}_{\pi'}(G)$. If x is a q-element of G, by hypothesis $|x^G|$ is a π-number, whence $T \subseteq C_G(x)$. Therefore, $x \in C_G(T) \subseteq T$, whence q does not divide $|G|$ and the thesis of the theorem is trivially true.

Accordingly, we will assume that $\mathbf{O}_\pi(G) > 1$. If H is any p-complement of G, by induction we have that $H\mathbf{O}_\pi(G)/\mathbf{O}_\pi(G) \cong H/H \cap \mathbf{O}_\pi(G)$ has a normal Sylow q-subgroup. As $H \cap \mathbf{O}_\pi(G)$ is a q-subgroup of H, we conclude that H has a normal Sylow q-subgroup too, as wanted. $\square$

In the next result, we show that the p-complements of G are indeed nilpotent when we add to the hypotheses of the above theorem the existence of some q-element in G whose index is the highest power of q dividing the sizes of classes in $\mathrm{Con}(G_{p'})$.

Theorem 6. *Let G be a finite p-solvable group and $\pi = \{p, q\}$ with q and p two distinct primes. Suppose that the sizes of classes in $\mathrm{Con}(G_{p'})$ are π-numbers. Let q^b be the highest power of the prime q which divides the sizes of classes in $\mathrm{Con}(G_{p'})$. Suppose that there exists some q-element $x \in G$ such that $|x^G| = q^b$. Then G has nilpotent p-complements.*

Proof. By Theorem 5 we know that G is solvable. Let Q be a Sylow q-subgroup of G with $x \in Q$. Since $G = QC_G(x)$, then $K := \langle x^g \mid g \in G \rangle = \langle x^g \mid g \in Q \rangle$ is a normal q-subgroup of G.

Let T be a π-complement of G with $T \subseteq C_G(x)$. For any $y \in T$, we have $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$ and then the hypotheses imply that $|C_G(x) : C_G(x) \cap C_G(y)|$ is a p-number. Therefore, $C_K(x) = C_K(x) \cap C_K(y) \subseteq C_K(y)$ and by applying Theorem 3, we obtain $C_K(y) = K$. If we write $Z := C_G(K)$, we have just proved that $T \subseteq Z \trianglelefteq G$, with $|G : Z|$ a π-number. Thus, if y is any π'-element of Z, then $|y^Z|$ is a π-number, so Lemma 4(b) implies that Z has abelian Hall π'-subgroups and $l_{\pi'}(Z) \leqslant 1$. Thus, we can write $R := \mathbf{O}_{\pi,\pi'}(Z) = \mathbf{O}_\pi(Z)T$ with T abelian.

Now, let P_0 be a Sylow p-subgroup of $\mathbf{O}_\pi(Z)$. By Frattini's argument, we can write $R = \mathbf{O}_\pi(Z)N_R(P_0)$. Moreover, without loss of generality we may suppose that $T \subseteq N_R(P_0)$. Now, for any $y \in T$, since $T \subseteq Z \subseteq C_G(x)$, we have $C_G(xy) = C_G(x) \cap C_G(y)$ and $|C_G(x) : C_G(xy)|$ is a p-number by hypothesis. As $R \subseteq C_G(x)$ then $|R : C_R(y)|$ is a p-number too, whence there exists some $Q_0 \in \mathrm{Syl}_q(R)$ such that $Q_0 \subseteq C_R(y)$. Hence, for any $y \in T$ we have $y \in C_T(Q_0)$ for some $Q_0 \in \mathrm{Syl}_q(R)$. Now, if $g \in TP_0$, then $g \in C_{TP_0}(Q_0)P_0$ for some $Q_0 \in \mathrm{Syl}_q(R)$. Since $R = P_0Q_0T$, then

$$TP_0 \subseteq \bigcup_{g \in TP_0} C_{TP_0}\big(Q_0^g\big)P_0 = \bigcup_{g \in TP_0} \big(C_{TP_0}(Q_0)P_0\big)^g,$$

which forces $TP_0 = C_{TP_0}(Q_0)P_0$. As T is a p-complement of TP_0, there exists some $g \in TP_0$ such that $T^g \subseteq C_{TP_0}(Q_0)$. Thus, $T \times Q_0^{g^{-1}}$ is a p-complement of R.

Now, choose a p-complement H of G such that $T \times Q_0^{g^{-1}} \subseteq H$. Since $R \trianglelefteq G$, we have that $H \cap R$ is a p-complement of R, so $H \cap R = T \times Q_0^{g^{-1}} \trianglelefteq H$. Therefore, $T \trianglelefteq H$. By applying Theorem 5, we conclude that H is nilpotent. □

Now we are ready to prove the particular case of our main theorem, that is, when the set of p-regular conjugacy class sizes is $\{1, p^a, q^b, p^aq^b\}$ for some prime q. This is a consequence of the following.

Theorem 7. *Let G be a p-solvable group whose conjugacy class sizes of p'-elements are $\{1, p^{a_1}, \ldots, p^{a_r}, q^b, p^{c_1}q^b, \ldots, p^{c_s}q^b\}$, where q is prime distinct from p and $c_i > 0$, $b, a_i \geqslant 0$ for all i. Then the p-complements of G are nilpotent.*

Proof. By Theorem 5 we know that G is solvable. If $b = 0$, we use Lemma 4(b) with $\pi = \{p\}$ to obtain that G has abelian p-complements, so we may suppose that $b > 0$.

If there exists some q-element of index q^b, then Theorem 6 applies and the p-complements of G are nilpotent, so the theorem is proved. Suppose now that there exists some q-element $x \in G$ such that $|x^G| = p^{c_i}q^b$ for some i with $c_i \geqslant 0$ and let T be a $\{p, q\}$-complement of G, which is abelian by Lemma 4(b), and such that $T \subseteq C_G(x)$. Let Q_0T be a p-complement of $C_G(x)$, where $Q_0 \in \mathrm{Syl}_q(C_G(x))$ and $x \in Q_0$. Now, if $y \in T$ then $C_G(xy) = C_G(x) \cap C_G(y)$ and notice that the hypotheses of the theorem imply that $|C_G(x) : C_G(xy)|$ must be a p-power. On the other hand, $T \subseteq C_G(xy)$ since T is abelian, and consequently, there exists some $g \in C_G(x)$ such that TQ_0^g is a p-complement of $C_G(xy)$. Now, if we take a p-complement H of G such that $TQ_0^g \subseteq H$, we can certainly write $H = TQ$, for some $Q \in \mathrm{Syl}_q(G)$, with $Q_0^g \subseteq Q$. As $Q_0^g \subseteq C_Q(xy) \subseteq C_Q(x)$

and $Q_0^g \in \mathrm{Syl}_q(C_G(x))$, it follows that $C_Q(xy) = C_Q(x)$ and thus $C_Q(x) \subseteq C_Q(y)$. Furthermore, by Theorem 5, we know that $Q \trianglelefteq H$. Notice that $x \in Q$ and then we can apply Theorem 3 to conclude that $C_Q(y) = Q$, for all $y \in T$. Then $H = Q \times T$, and since T is abelian, we deduce that H is nilpotent and this case is finished too.

As a result, we may assume that every q-element of G is central in G or has p-power index. Choose $y \in G_{p'}$ such that $|y^G| = q^b$ and write $y = y_q y_{q'}$, where y_q and $y_{q'}$ are the q-part and q'-part of y, respectively. Since $C_G(y) \subseteq C_G(y_q)$, it follows that y_q must be central in G and thus, by replacing y by $y_{q'}$, we may assume that y is a $\{p, q\}'$-element. Let $H = QT$ be a p-complement of G, where $Q \in \mathrm{Syl}_q(G)$ and T is a $\{p, q\}$-complement of G with $y \in T$. As $G = QC_G(y)$, then

$$L = \langle y^g \mid g \in G \rangle = \langle y^q \mid g \in Q \rangle \subseteq H.$$

As L is normal in G, we deduce that $L \subseteq C_G(x)$ for any $x \in Q$. Then $Q \subseteq C_G(L) \subseteq C_G(y)$, and this is a contradiction. □

We remark that in Theorem 7 the case $c_i = 0$ for all i and $a_i > 0$ for some i was discussed by the authors in Theorem D of [2], showing that this case cannot happen.

Corollary 8. *Let G be a p-solvable group whose conjugacy class sizes of p'-elements are $\{1, r^a, q^b, r^a q^b\}$, where q and r are primes (distinct from or equal to p) and a and b positive integers. Then the p-complements of G are nilpotent.*

Proof. When $r \neq p \neq q$ we apply Lemma 1 and Camina's theorem and obtain that any p-complement of G is nilpotent. Otherwise, we apply Theorem 7. □

3. Proof of Theorem A

In order to prove Theorem A, we will need the following lemma and a classical result due to Itô, which characterizes the structure of those groups which possess only two conjugacy class sizes.

Lemma 9. *Let G be a finite p-solvable group and let $\pi = \{p, q\}$. Suppose that the size of any conjugacy class of π'-elements is a p-number. Then G is solvable, the π-complements of G are abelian and each p-complement of G has a normal (abelian) q-complement.*

Proof. We show first that G is solvable by induction on $|G|$. If $\mathbf{O}^p(G) < G$, then as the hypotheses are inherited by normal subgroups, it clearly follows that G is solvable. Therefore, we will assume that $\mathbf{O}^p(G) = G$ and hence, $\mathbf{O}^{p'}(G) < G$. Notice that if $\mathbf{O}^{p'}(G) = 1$, then G is a p'-group and the hypotheses imply that every q'-element of G is central, so G is trivially solvable. We can assume then that $\mathbf{O}^{p'}(G) > 1$ and write $\overline{G} = G/\mathbf{O}^{p'}(G)$. It is easy to see that the hypotheses are inherited by factor groups and we will prove it for $\overline{G}$. Let $\bar{x} \in \overline{G}$ a π'-element and factor $x = x_\pi x_{\pi'}$ where x_π and $x_{\pi'}$ are the π-part and π'-part of x, respectively. Then $\bar{x} = \overline{x_{\pi'}}$, so x can be assumed without loss to be a π'-element. As $|\bar{x}^{\overline{G}}|$ divides $|x^G|$, we obtain that the class size of any π'-element of $\overline{G}$ is also a p-number, as wanted. By applying the inductive hypothesis to $\mathbf{O}^{p'}(G)$ and $\overline{G}$, we conclude that G is solvable as wanted.

The fact that the π-complements of G are abelian follows just by applying Lemma 4(b).

We prove now by induction on $|G|$ that each p-complement of G, say H, has a normal q-complement. Let $N := \mathbf{O}_{q'}(G)$ and suppose first that $N = 1$. Since the index of any π'-element $y \in G$ is a p-number, then $\mathbf{O}_q(G) \subseteq C_G(y)$ and so $y \in C_G(\mathbf{O}_q(G)) \subseteq \mathbf{O}_q(G)$, which is a contradiction. Therefore, in this case there are no π'-elements in G, that is, G is a $\{p, q\}$-group and the conclusion of the theorem is trivial. Hence, we will assume that $N > 1$ and apply the inductive hypothesis to G/N so as to obtain that $HN/N \cong H/H \cap N$ has a normal q-complement. As $H \cap N$ is a q'-subgroup, it follows that H also has a normal q-complement. □

Theorem 10. *Suppose that* 1 *and* $m > 1$ *are the only lengths of conjugacy classes of a group* G. *Then* $G = P \times A$, *where* $P \in \mathrm{Syl}_p(G)$ *and* A *is abelian. In particular, then* m *is a power of* p.

Proof. See Theorem 33.6 of [7]. □

Proof of Theorem A. We will show that m is a power of some prime $q \neq p$ and then the result will be proved by Corollary 8. First, we show that two p'-elements of index p^a and m centralize each other.

Step 1. If w is a p'-element of index m and y is a p'-element of index p^a, then $w \in C_G(y)$.

Proof. Let H be a p-complement of G with $w \in H$. Notice that $G = C_G(w)H$ and that there exists some $g \in H$ such that $w^g \in H^g \subseteq C_G(y)$. Also, as w and y have coprime index, we have $G = C_G(w)C_G(y)$, so we can assume that $g \in C_G(w)$. Thus, $w = w^g \in H^g \subseteq C_G(y)$. □

There exist p'-elements of index p^a by hypothesis, so by considering the primary decomposition of such elements, there must exist certain q-elements of index p^a for some prime q. For any such a prime q we prove the following properties (Steps 2–4).

Step 2. For any p-complement H of G, it holds that every q'-element of H has index 1 or m in H.

Proof. Since the centralizer of any q-element of index p^a contains some p-complement of G, by conjugacy we may certainly choose some q-element, say y, of index p^a such that $H \subseteq C_G(y)$. Now, let z be any q'-element of H which centralizes y and then $C_G(zy) = C_G(z) \cap C_G(y) \subseteq C_G(y)$. Thus, z has necessarily index 1 or m in $C_G(y)$. As m is a p'-number, it follows that $C_G(y) = H(C_G(z) \cap C_G(y))$ and we conclude that $|H : C_H(z)| = 1$ or m as required. □

Step 3. If z is a q-element of index $p^a m$, then $C_G(z) = Q_z P_z \times T_z$, where Q_z and P_z are q and p-subgroups respectively and T_z is an abelian $\{p, q\}'$-subgroup. Furthermore, if z lies in some p-complement H of G, then we can assume that T_z is not central in H.

Proof. By the maximality of the index of z we notice that any $\{p, q\}'$-element $t \in C_G(z)$ satisfies $C_G(zt) = C_G(z)$, so $C_G(z) \subseteq C_G(t)$ and accordingly $C_G(z)$ can be written as described in the statement. We remark that this part of the step is also true without the assumption of existence of q-elements of index p^a that we are doing.

For the second part, we take some p-complement H of G with $z \in H$ and we will prove that if T_z is central in H, then the theorem is proved. Suppose that $T_z \subseteq \mathbf{Z}(H)$, and consequently, $T_z = \mathbf{Z}(H)_{q'}$ and

$$m_{q'} = |G|_{\{p,q\}'} / \bigl|\mathbf{Z}(H)\bigr|_{q'}. \tag{I}$$

We distinguish three cases. Assume first that H possesses a q'-element of index m in G, say w, and take a q-element $y \in H$ of index p^a. We know by Step 1 that $y \in C_G(w)$, so $C_G(wy) = C_G(w) \cap C_G(y)$ and certainly wy has index $p^a m$. On the other hand, (I) implies that $\mathbf{Z}(H)_{q'}$ is a $\{p,q\}$-complement of $C_G(wy)$. But as $w \in C_G(wy)$, then $w \in \mathbf{Z}(H)$ and this contradicts the fact that w has index m. Therefore, this case cannot happen. Assume now that there exists in H a q'-element of index $p^a m$. Again (I) shows that $\mathbf{Z}(H)_{q'}$ is a $\{p,q\}'$-complement of $C_G(w)$, and as in the above paragraph, this leads to a contradiction.

Finally, we can assume that any q'-element $w \in H$ has index p^a. This means that the class size of any $\{p,q\}'$-element of G is a p-number, so by applying Lemma 9, we have that G is solvable and the $\{p,q\}$-complements are abelian. Let s be any prime distinct from p and q and let $S \in \mathrm{Syl}_s(G)$ with $S \subseteq H$. If $S \subseteq \mathbf{Z}(H)$, it is trivial that s does not divide m. If $S \nsubseteq \mathbf{Z}(H)$, then we take $w \in S - \mathbf{Z}(H)$ and by Step 2, w has index m in H. As S is abelian, we have $S \subseteq C_H(w)$, so in particular, s does not divide m either. Therefore, m is a q-power and the theorem is proved. □

Step 4. If z is a q-element of index $p^a m$, lying in some p-complement H of G, then $|H : C_H(z)| = m$ and there exists some element $t \in T_z \cap H - \mathbf{Z}(H)$, where T_z is the subgroup defined in Step 3.

Proof. We show first that $|H : C_H(z)| = m$. Write $C_G(z) = Q_z P_z \times T_z$ as in Step 3, with T_z non-central in H and choose a non-central $\{p,q\}'$-element $w \in T_z$. This can be assumed of prime power order, say for instance an r-element for a prime $r \neq p, q$. We will distinguish three cases depending on the index of w in G.

Assume first that w has index p^a. In this case, Step 2 asserts that any r'-element of H, in particular z, has index m in H.

Suppose now that w has index m. Observe that any r'-element of $C_G(w)$ has index 1 or p^a in $C_G(w)$, so by Lemma 9, the $\{p,r\}$-complements of $C_G(w)$ are abelian. On the other hand, as T_z is abelian and $w \in T_z$, we have $C_G(z) \subseteq C_G(w)$, so in particular, Q_z is abelian. We have just shown that any p-complement of $C_G(z)$ is abelian. Moreover, as $|C_G(w) : C_G(z)| = p^a$, then the p-complements of $C_G(z)$ are also p-complements of $C_G(w)$. On the other hand, the fact that $HC_G(w) = G$ implies that $C_H(w)$ is an (abelian) p-complement of $C_G(w)$. Then $C_H(w) \subseteq C_H(z)$ and hence, $C_H(w) = C_H(z)$. Since by Step 2, w has index 1 or m in H, we conclude that z has the same index. But we notice that z cannot be central in H since it has index $p^a m$ in G. So this case is finished too.

Finally, assume that w has index $p^a m$. As T_z is abelian, then $C_G(z) \subseteq C_G(w)$ and by orders, $C_G(z) = C_G(w)$. Taking into account the decomposition of $C_G(z)$ and of $C_G(w)$ given in Step 3 (w is an r-element), we get

$$C_G(z) = C_G(w) = Q_z \times P_z \times T_z$$

with the same notation given there. On the other hand, we can take a q-element $y \in H$ of index p^a in G such that $y \in \mathbf{Z}(H)$. Since w has index $p^a m$ we have $C_G(wy) = C_G(w) \cap C_G(y) =$

$C_G(w)$, that is, $C_G(w) \subseteq C_G(y)$ and $|C_G(y) : C_G(w)| = m$. This forces $C_G(y) = HC_G(w)$ and accordingly, $|H : C_H(w)| = m$. Therefore, z also has index m in H, as we wanted to prove.

We prove now the second part of the step. By the first part we have $m_{q'} = |H|_{q'}/|C_H(z)|_{q'}$, but if we consider the decomposition of $C_G(z) = Q_z P_z \times T_z$, we also obtain $m_{q'} = |G_{\{p,q\}'}|/|T_z|$. Thus, $|T_z| = |C_H(z)|_{q'}$. If $T_z \cap H \subseteq \mathbf{Z}(H)$, then $T_z \cap H \subseteq \mathbf{Z}(H)_{q'} \subseteq C_H(z)_{q'}$. But, on the other hand, $C_H(z)_{q'}$ is clearly contained in the Hall $\{p,q\}'$-subgroup of $C_G(z)$, that is, in T_z. We deduce that $C_H(z)_{q'} = T_z$ and this forces T_z to be central in H, which is a contradiction by Step 3. □

It is clear that in G there exist elements of index m and prime-power order. From now on we will fix one of these elements x, an r-element, with $r \neq p$, and will choose a p-complement of G, say H, such that $x \in H$. Since $G = HC_G(x)$, then $C_H(x)$ is a p-complement of $C_G(x)$ and by applying Lemma 9 to $C_G(x)$, we can write $C_H(x) = T_x R_x$, with R_x an r-subgroup and T_x an abelian $\{p,r\}'$-subgroup which is normal in $C_H(x)$.

We know by Step 1 that any p'-element of index p^a commutes with any p'-element of index m, so in particular, every p'-element of index p^a of H belongs to $C_H(x)$. Now, in the two following steps we prove two properties related to $C_H(x)$.

Step 5. We may assume that T_x is not central in G.

Proof. We assume that $T_x \subseteq \mathbf{Z}(G)$ and work to get a contradiction. We know that every r'-element in H of index 1 or p^a centralizes x, and consequently, lies in T_x. Thus, there are not r'-elements in H of index p^a, whence there cannot exist such elements in G. Accordingly, there must exist some r-element y of index p^a, which can be assumed to lie in $\mathbf{Z}(H)$ by conjugacy.

By Step 2, any r'-element of H has index 1 or m in H. Notice that if any r'-element of H lies in $\mathbf{Z}(H)$, then $H = R \times \mathbf{Z}(H)_{r'}$, which is nilpotent and m would be a power of r. In this case the theorem is proved, so we can assume the existence of some r'-element $y \in H - \mathbf{Z}(H)$ of index m in H. Then we have $m_{r'} = |H|_{r'}/|C_H(y)|_{r'}$. But moreover, the structure of $C_G(x)$ provides the equality $m_{r'} = |H|_{r'}/|T_x|$. As $T_x = (\mathbf{Z}(G) \cap H)_{r'}$, this yields

$$|T_x| = \left|\mathbf{Z}(G) \cap H\right|_{r'} = \left|C_H(y)\right|_{r'}$$

and consequently, $(\mathbf{Z}(G) \cap H)_{r'}$ is an r-complement of $C_H(y)$. This contradicts the fact that y is non-central in $C_H(y)$. □

Step 6. If T_x has an element of index m or $p^a m$, then $C_H(x)$ is abelian.

Proof. Suppose first that there is an element $w \in T_x$ of index m. By considering the primary decomposition of w we can assume without loss that w is a q-element for some prime $q \neq p, r$, since $C_G(w)$ must be equal to the centralizer of some primary component of w. Notice that each $\{p,q\}'$-element of $C_G(w)$ has index 1 or p^a in $C_G(w)$, and that $C_H(w)$ is a p-complement of $C_G(w)$. So by applying Lemma 9, we can write $C_H(w) = Q_w T_w$ where Q_w is a q-subgroup and T_w an abelian $\{p,q\}'$-subgroup which is normal in $C_H(w)$. We also notice that $|C_H(w)| = |C_H(x)|$, as both subgroups have index $p^a m$ in G. We will prove that in fact both centralizers are equal. As $w \in T_x$, we have that $T_x \subseteq C_H(w)$. On the other hand, x lies in some Sylow r-subgroup of $C_H(w)$, say R_w, which is abelian, so $R_w \subseteq C_H(x)$. Therefore, $T_x R_w \subseteq C_H(x)$ and

$T_x R_w \subseteq C_H(w)$. By order considerations, we conclude $C_H(w) = C_H(x) = T_x R_w$. But we know that R_w is abelian and normal in $C_H(w)$, so $C_H(x) = T_x \times R_w$, whence $C_H(x)$ is abelian.

Suppose now that there is an element $w \in T_x$ of index $p^a m$. Notice that any r-element $z \in C_G(w)$ satisfies $C_G(zw) = C_G(w) \cap C_G(z) = C_G(w)$ by the maximality of the index of w. This means that z is central en $C_G(w)$, so we can write $C_G(w) = T_w P_w \times R_w$, with P_w a p-subgroup, T_w a $\{p, r\}'$-subgroup and R_w an abelian r-subgroup. Moreover, as T_x is abelian, then $T_x \subseteq C_G(w)$, so T_x centralizes R_w. On the other hand, $x \in C_G(w)$, so $x \in R_w$ and $R_w \subseteq C_H(x)$. By order considerations R_w is a Sylow r-subgroup of $C_H(x)$, whence we conclude that $C_H(x) = R_w \times T_x$ and $C_H(x)$ is abelian too. □

In the rest of the proof we are going to define and work with certain subgroup L_q for any prime $q \neq p, r$. When these subgroups are central for all q we will define and work with a subgroup associated to r.

Step 7. For every prime $q \neq p, r$, let

$$L_q = \langle y\colon\ y \text{ is a } q\text{-element in } H \text{ with } |y^G| = 1 \text{ or } p^a \rangle.$$

Then L_q is an abelian normal q-subgroup of H.

Suppose that $L_q \subseteq \mathbf{Z}(G)$ for all $q \neq p, r$. Then we define

$$L_r = \langle y\colon\ y \text{ is an } r\text{-element in } H \text{ with } |y^G| = 1 \text{ or } p^a \rangle$$

and it is a non-central abelian normal r-subgroup of H. Furthermore, in this case $C_H(x)$ is abelian.

Proof. For any prime $q \neq p, r$, we know that $C_H(x)$ has an abelian normal Sylow q-subgroup Q. Likewise, we know by Step 1 that

$$\{y \in H\colon\ y \text{ is a } q\text{-element with } |y^G| = 1 \text{ or } p^a\} \subseteq C_H(x).$$

As a consequence, $L_q \subseteq Q$ and thus, L_q is an abelian q-subgroup of H. The fact that $L_q \trianglelefteq H$ is trivial.

Suppose now that $L_q \subseteq \mathbf{Z}(G)$ for all $q \neq p, r$. This implies that there are no q-elements of index p^a for all such primes and hence, there must be an r-element in G (and in H) of index p^a. In particular, $L_r \nsubseteq \mathbf{Z}(G)$. On the other hand, by applying Step 5, we deduce that in T_x there must be elements of index m or $p^a m$, so by Step 6, $C_H(x)$ is abelian. But by Step 1, we have

$$\{y \in H\colon\ y \text{ is an } r\text{-element with } |y^G| = 1 \text{ or } p^a\} \subseteq C_H(x),$$

so L_r is contained in the Sylow r-subgroup of $C_H(x)$. Therefore, L_r is an abelian r-subgroup of H which is trivially normal in H. □

Step 8. Every q-element of H centralizes L_q for any prime $q \neq p, r$. If $L_q \subseteq \mathbf{Z}(G)$ for any $q \neq p, r$, then any r-element of H centralizes L_r.

Proof. Let s be any prime distinct from p and let z be an s-element of H. We will prove that $z \in M := C_H(L_s)$ (we remark that when $s = r$ then we are assuming that $L_q \subseteq \mathbf{Z}(G)$ for all $q \neq p, r$).

If z has index p^a, then by definition $z \in L_s$, so trivially $z \in M$. If z has index m, we know by Step 1 that z centralizes any element of index p^a, so z also lies in M.

Thus, we only have to show that if z has index $p^a m$, then it lies in M too. By Step 3, we write $C_G(z) = S_z P_z \times T_z$ with the notation given there and T_z abelian. Also, by Step 4, there exists some $t \in T_z \cap H - \mathbf{Z}(H)$, so we have $C_G(z) \subseteq C_G(t)$. In particular $C_{L_s}(z) \subseteq C_{L_s}(t)$, and by applying Theorem 3, we conclude that $t \in M$. Now we distinguish three cases for the index of t in G. If t has index $p^a m$, then $C_G(t) = C_G(z)$, so z lies trivially in M. If t has index p^a, as t is non-central in H then by Step 2 (notice that there are s-elements of index p^a), we get $|H : C_H(t)| = m$. On the other hand, by Step 4, we have $|H : C_H(z)| = m$, and since $C_H(z) \subseteq C_H(t)$ we obtain by order considerations that $C_H(z) = C_H(t)$. It follows that $z \in M$. Finally, suppose that t has index m. There is no loss if we assume that t is an l-element, for some prime $l \neq s, p$, since we can replace t by some of its components in the primary decomposition, with the same index m. By applying Lemma 9 to $C_G(t)$, we get that $G_G(t)$ has abelian $\{p, l\}$-complements, so $C_H(t)$, which is a p-complement of $C_G(t)$, has an abelian normal s-complement, say T_t. Then $z \in T_t$ and $T_t \subseteq C_H(z)$. Therefore, $|C_H(t) : C_H(z)|$ is an l-number. As L_s is a normal s-subgroup of $C_H(t)$, we conclude that $L_s \subseteq C_H(z)$. □

Step 9. We can assume that for any prime $q \neq p, r$, we have $L_q \subseteq \mathbf{Z}(H)$. If $L_q \subseteq \mathbf{Z}(G)$ for all prime $q \neq p, r$, then $L_r \subseteq \mathbf{Z}(H)$.

Proof. Let s be a prime distinct from p. Notice that $L_s \subseteq C_H(x)$ by Step 1. We will consider the following cases:

(a) $s = r$. In this case notice that we are assuming by definition of L_r in Step 7 that $L_q \subseteq \mathbf{Z}(G)$ for all prime $q \neq p, r$. Also in this case, as T_x is non-central by Step 5 and there are no $\{p, r\}'$-elements of index p^a, then T_x has elements of index m or $p^a m$ and by Step 6 we have that $C_H(x)$ is abelian.

(b) $s \neq p, r$. We will distinguish two possibilities:
   (1) there are no r-elements of index p^a; and
   (2) there are r-elements of index p^a.

In cases (a) and (b)(1) we will see that if $w \in H$, then $w \in C_H(L_s)$, so $L_s \subseteq \mathbf{Z}(H)$. In case (b)(2) we have by Step 2 that every r'-element of H has index 1 or m in H. We will prove that if $w \in H$, then $w \in R_x^g C_H(L_s)$ with $g \in C_H(x)$. Once this is proved, we have

$$H = \bigcup_{g \in H} R_x^g C_H(L_s),$$

which forces that $H = R_x C_H(L_s)$ and $|H : C_H(L_s)|$ is an r-number. Let $y \in L_s - \mathbf{Z}(H)$, then $C_H(L_s) \subseteq C_H(y) \subseteq H$ and $|H : C_H(y)| = m$. Thus m is a r-power and the theorem would be proved. Therefore, $L_s \subseteq \mathbf{Z}(H)$ as we want to prove.

Now we prove the properties stated in the above paragraph. Let $w \in H$ and consider the $\{s, s'\}$-decomposition of w. By Step 8 we know that the s-part of w lies in $C_H(L_s)$, so we may suppose that w is an s'-element. If w has index m, then $w \in C_H(L_s)$ by Step 1. So we will study the cases in which w has index p^a or mp^a.

Suppose first that w has index p^a. Using Step 1 again we get $w \in C_H(x)$. In case (a) we know that $C_H(x)$ is abelian and $L_r \subseteq C_H(x)$, so clearly $w \in C_H(L_r)$. In case (b), we have $s \neq r$ and thus, $L_s \subseteq T_x$. We consider the $\{r, r'\}$-decomposition of $w = w_r w_{r'}$, so $w_{r'} \in T_x$ by Step 1. Since T_x is abelian, we obtain $w_{r'} \in C_H(L_s)$. In case (b)(1), w_r is central in G, so $w \in C_H(L_s)$. In case (b)(2), as $w_r \in C_H(x) = T_x R_x$, then $w_r \in R_x^g$, for some $g \in C_H(x)$. So $w \in R_x^g C_H(L_s)$.

Suppose now that w has index $p^a m$ and consider the primary decomposition of w, that is, $w = w_r w_{r'} = w_r w_{q_1} w_{q_2} \cdots w_{q_k}$ for some primes q_i. If w_{q_i} for all i has index p^a or m then, by the above paragraphs, we have $w_{r'} \in C_H(L_s)$. On the other hand, if w_r has index m, then $w_r \in C_H(L_s)$ by Step 1 and $w \in C_H(L_s)$. If w_r has index p^a then, again by the above paragraph, we deduce in cases (a) and (b)(1) that $w_r \in C_H(L_s)$ and, in case (b)(2), we obtain $w_r \in R_x^g C_H(L_s)$, so $w \in R_x^g C_H(L_s)$, for some $g \in C_H(L_s)$. Thus, we can assume that either w_r or w_{q_i}, for some prime q_i, has index $p^a m$ in G. We will prove that $w \in C_H(L_s)$ in all cases (a), (b)(1) and (b)(2).

Let us consider w_l (with l either equal to q_i or r) such that $|w_l^G| = p^a m$. Notice that $l \neq s$ and $C_G(w) = C_G(w_l)$. Suppose that $L_s \nsubseteq \mathbf{Z}(H)$ and take $y \in L_s - \mathbf{Z}(H)$. As y has index p^a, by conjugacy we can assume without loss that $H \subseteq C_G(y)$. Then w centralizes y and $C_G(wy) = C_G(w) \cap C_G(y) = C_G(w) = C_G(w_l) \subseteq C_G(y)$. Since $C_G(y) = HC_G(w_l)$, it follows that $C_H(w_l)$ is a p-complement of $C_G(w_l)$. On the other hand, arguing in a similar way as in the first part of Step 3, we get $C_G(w_l) = L_{w_l} P_{w_l} \times A_{w_l}$, with L_{w_l} an l-group, P_{w_l} a p-group and A_{w_l} an abelian $\{p, l\}'$-group, whence $y \in A_{w_l}$. As $L_{w_l} \times A_{w_l}$ is also a p-complement of $C_G(w_l)$, then we can assume up to conjugacy that $C_H(w_l) = L_{w_l} \times A_{w_l} \subseteq H$. Thus $m_s = |G|_s / |A_{w_l}|_s \leqslant |G|_s / |\mathbf{Z}(H)|_s$. If $|A_{w_l}|_s = |\mathbf{Z}(H)|_s$, then $|G|_s / |\mathbf{Z}(H)|_s = m_s$. Moreover, as $L_s \subseteq C_G(x)$, then $|G|_s / |L_s| \geqslant m_s$. Since $\mathbf{Z}(H)_s \subseteq L_s$, we obtain $L_s = \mathbf{Z}(H)_s$, which contradicts our assumption. So we can assume that there are s-elements in A_{w_l} which are not central in H. We distinguish the following three cases.

Suppose first that there is an s-element $z \in A_{w_l}$ of index $p^a m$. Then $C_G(z) = C_G(w_l) = C_G(w)$. By Step 7, we have that $z \in C_H(L_s)$ and then $L_s \subseteq C_G(z) = C_G(w)$. In this case, we obtain $w \in C_G(L_s)$.

Suppose now that there is an s-element $z \in A_{w_l}$ of index m. As $z, y \in A_{w_l}$, then $C_G(w_l) = C_G(w) \subseteq C_G(y) \cap C_G(z) = C_G(yz)$ and thus, $C_G(yz) = C_G(w)$. Again by Step 7, $L_s \subseteq C_H(z) = C_H(w)$, so $w \in C_H(L_s)$.

Finally, assume that every s-element of A_{w_l} has index p^a, and consequently, that all of them belong to L_s. Then $|A_{w_l}|_s \leqslant |L_s|$, so $m_s \leqslant |G|_s / |L_s| \leqslant |G|_s / |A_{w_l}|_s = m_s$. Therefore, $L_s \subseteq A_{w_l}$. As A_{w_l} is abelian, we conclude that $w_l \in A_{w_l} \subseteq C_H(L_s)$, whence $L_s \subseteq C_H(w_l) = C_H(w)$ and $w \in C_G(L_s)$ as we wanted to prove. □

We remark that whenever there exists some s-element of index m in G for some prime $s \neq p$, then Step 9 also holds for s, just by arguing with s instead of r as we have made it in Steps 5–9.

Step 10. Conclusion.

Proof. We know that there are p'-elements of index p^a, so there is some prime $s \neq p$ such that $L_s \nsubseteq \mathbf{Z}(G)$. By Step 9, we have $L_s \subseteq \mathbf{Z}(H)$. Notice that if $s = r$, then we are also assuming that $L_q \subseteq \mathbf{Z}(G)$ for all prime $q \neq p, r$. We claim first that any s-element $w \in H$ has index 1 or m in H. We distinguish three possibilities according to the index of w in G. If w has index p^a, then $w \in L_s \subseteq \mathbf{Z}(H)$. If w has index m, then $HC_G(w) = G$ and it clearly follows that w has index m

in H. Finally, if w has index $p^a m$ in G, then by Step 4, w has index m in H, and the claim is proved.

On the other hand, by Step 2, we also have that any s'-element of H has index 1 or m in H. The rest of the proof consists of showing that any element of H has index 1 or m too. Then, by applying Theorem 10, we get that H is nilpotent and m is a prime power, so the proof of the theorem will be finished.

Let us take any $z \in H$ and factor $z = z_s z_{s'}$. If one of these factors is central in H, then z would have the same index in H as the other factor, and consequently, z would have index 1 or m in H. Therefore, we will assume that both z_s and $z_{s'}$ are not central in H. We distinguish three cases for the index of z in G. If z has index p^a in G, then as $C_G(z) \subseteq C_G(z_s)$ and since z_s cannot be central in G, it follows that $C_G(z) = C_G(z_s)$, whence z has the same index in H as z_s, that is, m. If z has index m in G, since $HC_G(z) = G$, we easily deduce that z has index m in H too. Thus, we will suppose that z has index $p^a m$ in G.

We have the following possibilities for the index of z_s in G. If z_s has index p^a, then it would be central in H by Step 9, but we are assuming that it is not so. If z_s has index $p^a m$ in G, we certainly have that $C_G(z) = C_G(z_s)$ and then z has the same index in H as z_s. We will assume then that z_s has index m in G.

On the other hand, we analyze the index of $z_{s'}$ in G. Suppose first that $z_{s'}$ has index p^a. If we consider the decomposition of $z_{s'} = z_{q_1} \ldots z_{q_t}$, it is clear that z_{q_i} has index p^a or 1, whence if $q_i \neq r$ for all i, then by Step 9, $z_{q_i} \in \mathbf{Z}(H)$ for all i. Hence, $z_{s'}$ is central in H, contradicting our assumption. Therefore, we can assume that there is some i such that $q_i = r$ and that $z_{s'} = z_r y$ with $y \in \mathbf{Z}(H)$. As $r \neq s$ and z_s has index m, by the remark above this step, we know that Step 9 holds for s, that is, any s'-element of index p^a is central in H. In particular, $z_r \in \mathbf{Z}(H)$ and consequently, $z_{s'} \in \mathbf{Z}(H)$ too, which is a contradiction.

Suppose now that $z_{s'}$ has index $p^a m$. Then $C_G(z_{s'}) = C_G(z)$, so z has index 1 or m in H.

Finally, let us assume that $z_{s'}$ has index m and consider again the primary decomposition of $z_{s'}$ as above. It follows that $C_G(z_{s'}) = C_G(z_l)$ for some prime $l \neq p, r$. Then

$$C_G(z) = C_G(z_s) \cap C_G(z_{s'}) = C_G(z_s) \cap C_G(z_l) = C_G(z_s z_l)$$

and accordingly we can assume that $z = z_s z_t$, knowing that both factors have index m in G. Now, by applying Lemma 9, we have that $C_G(z_s)$ has abelian $\{p, s\}$-complements, so we can write $C_H(z_s) = T_0 S_0$, where S_0 is an s-subgroup and T_0 an abelian $\{p, s\}'$-subgroup, with $z_t \in T_0$. Notice that $T_0 \subseteq C_H(z_l)$. Arguing in the same way with $C_G(z_l)$, it has abelian $\{p, l\}$-complements, so in particular we may write $C_H(z_l) = T_1 S_1$, where T_1 is a $\{p, s\}'$-subgroup and S_1 is an abelian s-subgroup, with $z_s \in S_1$. Notice that $S_1 \subseteq C_H(z_s) = T_0 S_0$. Also, up to conjugacy and by order considerations we can assume that $S_1 = S_0$, so $C_H(z_s) = S_0 T_0 \subseteq C_H(z_l)$. As both subgroups have the same order, we conclude that $C_H(z_s) = C_H(z_l)$. Therefore,

$$C_H(z) = C_H(z_s) \cap C_H(z_l) = C_H(z_l) = C_H(z_s),$$

whence z has index m in H, as we wanted to prove. □

As a consequence of Theorem A and the main result of [5], we obtain the following

Corollary 11. *Let G be a p-solvable group whose set of conjugacy class sizes of all p'-elements of G is $\{1, m, n, mn\}$, with m and n positive integers such that $(m, n) = 1$, and either p does not* *divide m and n, or $n = p^a$. Then the p-complements of G are nilpotent and m and n are prime powers.*

Proof. Suppose that the set of conjugacy class sizes of all p'-elements of G is $\{1, m, n, mn\}$, with m and n positive integers such that $(m, n) = 1$, and that p does not divide m and n. By Lemma 1, we have $G = P \times H$, where H is a p-complement of G. Then the set of class sizes of H is $\{1, m, n, mn\}$ and Corollary B of [5] asserts that H is nilpotent and m and n are prime powers.

The other possibility, that is, when $n = p^a$, is exactly Theorem A. □

References

[1] M. Aschbacher, Finite Group Theory, Cambridge Univ. Press, New York, 1986.

[2] A. Beltrán, M.J. Felipe, Certain relations between p-regular class sizes and the p-structure of p-solvable groups, J. Austral. Math. Soc. 77 (2004) 1–14.

[3] A. Beltrán, M.J. Felipe, Prime powers as conjugacy class lengths of π-elements, Bull. Austral. Math. Soc. 69 (2004) 317–325.

[4] A. Beltrán, M.J. Felipe, Variations on a theorem by Alan Camina on conjugacy class sizes, J. Algebra 296 (2006) 253–266.

[5] A. Beltrán, M.J. Felipe, Some class size conditions implying solvability of finite groups, J. Group Theory, in press.

[6] A.R. Camina, Arithmetical conditions on the conjugacy class numbers of a finite group, J. London Math. Soc. (2) 5 (1972) 127–132.

[7] B. Huppert, Character Theory of Finite Groups, de Gruyter Exp. Math., vol. 25, de Gruyter, Berlin, 1998.